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LETTER TO THE EDITOR

Relaxation of hierarchical models defined on Sierpinski gasket fractals

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Abstract. In this letter diffusion in the presence of a hierarchical array of barriers is defined on the Sierpinski gasket and is generalised to d dimensions. Through an exact renormalisation group procedure the anomalous long-time behaviour of the autocorrelation function is obtained and compared with that of a similar hierarchical diffusion on Euclidean d-dimensional lattices. Some evidence of a transition from anomalous to normal behaviour is presented.

Anomalous relaxation, characteristic of processes with different timescales, occurs in very different physical systems ranging from molecular diffusion [1] to spin glasses [2]. Since the suggestion of Palmer *et al* [3] that this anomalous behaviour may be described in the context of hierarchical constrained dynamics, some stochastic models have been proposed and analysed [4, 5]. The model studied by Huberman and Kerszberg [5] consists in the hopping of a particle in a one-dimensional chain with a hierarchical array of barriers and is called the 1D ultradiffusion model due to the fact that the diffusion has an ultrametric structure (which is also discussed in the other model related to spin glasses [4]).

In a recent paper [6] a transition was found in the dynamics of the 1D ultradiffusion model from normal to anomalous diffusion as the parameter $R = w_{i+1}/w_i$ is varied, where w_i $(i = 0, 1, \ldots)$ is the transition rate associated to the barrier i. More precisely, the exponent x for the long-time behaviour of the autocorrelation function $P_0 \sim t^{-x}$ is

$$\begin{aligned} &x = x(R) && 0 < R < R_c \\ &x = \tfrac{1}{2} && R_c < R < 1 \end{aligned} \tag{1}$$

where $X(R)$ is a continuously varying function associated with a line of fixed points which extends up to $R_c = \frac{1}{2}$. Moreover, it was found that this transition is dependent on the distribution of transition rates rather than on the hierarchical spatial arrangement of the barriers. However, the self-similar or fractal nature of the barriers in the ultradiffusion model is very convenient when a renormalisation group (RG) approach is used. In fact, the 1D model was exactly solved within a RG treatment [7] and the anomalous exponent was found to be

$$x = \ln 2/\ln[2(2w_1^* + w_0)/w_1^*]. \tag{2}$$

This expression is valid for arbitrary values of the transition rate w_i. For the particular set of parameters w_0, w_1, $w_{n-1} = Rw_n$ $(n \geq 1)$, following a procedure similar to that reported below, it can be shown that expression (2) coincides with that obtained by

Teitel and Domany [6] (equation (10)) but it is only valid for R less than $\frac{1}{2}$, which is the critical value.

In this letter, the hierarchical structure of the barriers is provided by the fractal nature of the Sierpinski gasket (SG) and its generalisations to d dimensions. Exact solutions can then be worked out with decimation techniques giving results which interpolate between those of 1D and higher-dimensional ones. Besides, the study of diffusion in SG is interesting in its own right since these fractals are considered as non-random models of the backbone of the infinite cluster at the percolation threshold [8] and of other phenomena which take place in spaces of non-integer (fractal) dimension.

The diffusion process studied in this letter is defined as follows. Suppose that a SG has been constructed by aggregation from triangles in the atomic scale up to infinity (figure 1). The basic triangles are considered as cells and the shaded regions as energy barriers labelled by i, the stage of the construction in which they appear. As before, the probability of a particle crossing the barrier i in the unit of time is denoted by w_i. Now consider a particle hopping from a cell to a nearest-neighbour one through the corresponding barrier. For example, as one can see in figure 2(a), the particle in cell A can hop to the cell B through a barrier w_0, from B to D through a barrier w_1, etc. Assigning a site to the centre of each cell one can devise an effective fractal which describes the diffusion in a more conventional way (figure 3). The behaviour of the autocorrelation function corresponding to normal diffusion in this effective lattice is identical with that on the SG. This result will be explicitly shown later on.

Let $P_A(t)$, $P_B(t)$, ..., be the probability of finding the particle at cell A, B, ..., at time t and $\tilde{P}_A(\lambda)$, $\tilde{P}_B(\lambda)$, ..., their Laplace transforms. Then the diffusion is described by an infinite set of equations of the type (see figure 3)

$$\lambda \tilde{P}_B = w_0(\tilde{P}_A - \tilde{P}_B) + w_0(\tilde{P}_C - \tilde{P}_B) + w_1(\tilde{P}_D - \tilde{P}_B). \tag{3}$$

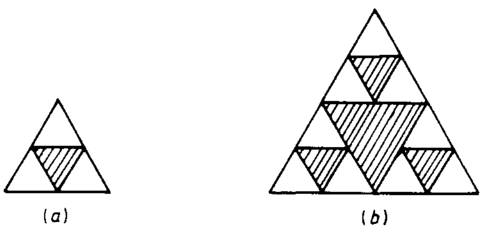

Figure 1. Construction of the SG by aggregation: (a) stage 0, (b) stage 1.

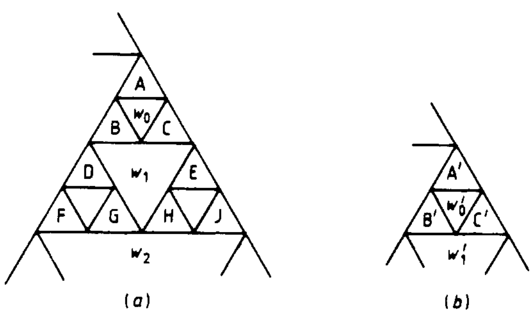

Figure 2. Cells and barriers in (a) the original fractal and (b) the renormalised one.

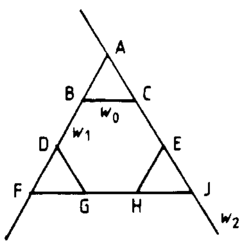

Figure 3. The effective lattice corresponding to figure 2(*a*). Some values of the transition rates are indicated.

Now we perform a decimation procedure which describes the sites B, C, D, E, G and H in terms of A, F and J. The 6×6 system can be reduced to a 3×3 one by imposing, in equation (3), solutions of the form

$$\tilde{P}_B = a\tilde{P}_A + b\tilde{P}_F + c\tilde{P}_J \qquad \tilde{P}_C = a\tilde{P}_A + c\tilde{P}_F + b\tilde{P}_J,$$

etc. Then the effective equations obtained for the surviving sites are recast in the form (3) with renormalised parameters which are given by the recursion relations

$$\lambda' = \gamma(w_0, w_1)\lambda \tag{4a}$$

$$\tilde{P}' = \alpha(w_0, w_1)\tilde{P} \tag{4b}$$

$$w_i' = \beta(w_0, w_1)w_{i+1} \qquad (i \geq 1) \tag{4c}$$

where

$$\alpha = \beta^{-1} = 3\gamma^{-1} = 3w_1/(3w_0 + 5w_1). \tag{4d}$$

These relations were obtained with the condition $w_0' = w_0$, which fixes the timescale, and for long-time behaviour, i.e. at lowest non-zero order in λ. Equation (4*c*) leads to a line of fixed points parametrised by w_1^* varying between 0 and $+\infty$. The fixed point $w_1^* = w_i^* = 0$ $(i \geq 2)$ describes the situation of trapping. The fixed point $w_1^* = w_i^* = +\infty$ $(i \geq 2)$ corresponds to a model with equal barriers inversely proportional to w_0, thus leading to ordinary diffusion. Note that in this case, B $\equiv$ D, C $\equiv$ E, G $\equiv$ H, etc, in figure 3, i.e. the effective fractal again reduces to the SG. By antitransforming (4*b*) and taking into account (4*a*) and (4*d*) one obtains

$$x = \ln 3/\ln[(3w_0 + 5w_1^*)/w_1^*] \tag{5}$$

with the limiting values $x = 0$ for $w_1^* = 0$ and $x = \ln 3/\ln 5$ for $w_1^* = +\infty$. As expected from the above comment, this value of the exponent x is twice the spectral dimension of the SG [9].

The diffusion studied above can be extended for the generalisations of the SG in d dimensions. One must consider each hypertetrahedron as a cell and assign a barrier to the regions limited by the cells. These barriers are again labelled as the stage of the fractal construction by aggregation. Then, by taking into account the geometrical symmetries of the d-dimensional SG, one can generalise equation (3) and, with a trick similar to that used in 2D, reduce the system of equations one obtains in place of (4*d*) and (5):

$$\alpha = \beta^{-1} = (d+1)\gamma^{-1} = (d+1)w_1/[(d+1)w_0 + (d+3)w_1] \tag{6a}$$

$$x = \ln(d+1)/\ln\{[(d+1)w_0 + (d+3)w_1^*]/w_1^*\}. \tag{6b}$$

In the limit $w_1^* = +\infty$ equation (6*b*) reduces to

$$x = \frac{\ln(d+1)}{\ln(d+3)} \tag{7}$$

which is the known exponent for the normal diffusion in a d-dimensional SG [9].

To this point no supposition about the values of the transition rates has been made. From equations (4*c*) and (4*d*) or (6*a*), it can be proved that the minimum parameter space closed under the RG transformation performed in this work is that defined by w_0 and w_1 with $w_i = R^{i-1}w_1$ ($i \geq 2$). That is, after a decimation step one recovers the same parameter set: w_0', w_1' and $w_i' = R^{i-1}w_1'$ ($i \geq 2$). Then from (4*c*) we obtain

$$w_1^* = w_0 R/\{1 - [(d+3)/(d+1)]R\} \tag{8}$$

which shows that equation (6*b*) breaks down at $R_d = (d+1)/(d+3)$. One can guess (recalling the comment made immediately after equation (2)) that at $R_c = R_d$ a transition to normal diffusion occurs.

By replacing (8) in (6*b*) one can obtain the R dependence of the anomalous exponent

$$x = \ln(d+1)/\ln[(d+1)/R] \tag{9}$$

which is an expression rather different from that advanced in [6] for the ultradiffusion in a d-dimensional Euclidean lattice:

$$x = d \ln 2/\ln(2/R). \tag{10}$$

Of course, both expressions are identical in $d = 1$.

Finally, we calculate the diffusion constant defined by

$$P_0(t) = (Dt)^{-x}. \tag{11}$$

By iterating the recursion relations (4*a*) and (4*b*) and by taking the Laplace transform of (11) one obtains

$$D = [\Gamma(1-x)]^{1/x}\left(\prod_{n=0}^{\infty} \gamma(w^{(n)})[\alpha(w^{(n)})]^{1-x}\right)^{1/x}. \tag{12}$$

From (4*d*) this expression reduces to

$$D = [\Gamma(1-x)]^{1/x}(\pi_0\pi_1)^{1/x} \tag{13}$$

where

$$\pi_0 = \prod_{n=0}^{\infty} \frac{(d+3)w_1^{(n)}}{(d+1)+(d+3)w_1^{(n)}} \tag{14a}$$

$$\pi_1 = \prod_{n=0}^{\infty} (d+1)^{(1/x)-(1/x_0)} \tag{14b}$$

with x and x_0 given by (6*b*) and (7) respectively. The transition rate w_0 has been set as one for simplicity. Through a straightforward procedure, for the set of parameters $w_0^{(n)} = 1$, $w_1^{(0)} = R$, $w_i^{(n)} = R^{i-1}w_1^{(n)}$ ($i \geq 2$), we obtain

$$\pi_0 = (R - R_\alpha)/R \tag{15}$$

for $R > R_d$, and zero otherwise. Moreover, $\pi_1 = 1$ ($R > R_d$) and $\pi_1 = 0$ ($R < R_d$). Then one has

$$D = [\Gamma(1-x)]^{1/x}(R - R_\alpha)/R \tag{16}$$

for $R > R_d$ (normal diffusion) and $D = 0$ for $R < R_d$ (anomalous diffusion). This result gives further support to the supposition that a transition from normal to anomalous relaxation occurs at $R = R_d = (d+1)/(d+3)$. Note that the R dependence of (16) is similar to that found by Teitel and Domany [6] for the one-dimensional hierarchical model.

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